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The Matrix Minimum Principle*

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The purpose of this paper is to provide an alternate statement of the Pontryagin maximum principle as applied to systems which are most conveniently and naturally described by matrix, rather than vector, differential or difference equations. The use of gradient matrices facilitates the manipulation of the resultant equations. The theory is applied to the solution of a simple optimization problem.

1. INTRODUCTION

The purpose of this paper is to provide (with no proofs) a statement of the necessary conditions for optimality for a class of problems that appear to be important, as evidenced by recent research efforts. This class of problems is distinguished by the fact that the plant equations are most conveniently described by matrix differential equations. For such problems, it is important to have a compact statement of the minimum principle so as to aid both intuition and mathematical manipulations; this provided the motivation for this paper.

In the remainder of this paper the following topics are treated:

(a) The relation of the matrix minimum principle to the ordinary minimum principle.

(b) A statement of the necessary conditions for optimality as provided by the matrix minimum principle.

(c) The solution of a very simple problem which involves the determination of the linear time-varying gains which optimize the response of a linear system with quadratic performance index.

The most common form of the minimum principle pertains to the optimal control of systems described by vector differential equations of the form

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$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] \tag{1}$$

(where $\mathbf{x}(t)$ is a column n-vector, $\mathbf{u}(t)$ is a column r-vector, and $\mathbf{f}(\cdot)$ is a vector-valued function). These are the type of systems considered by Pontryagin *et al.* (see reference *1*) and treated in most of the available books dealing with modern control theory. The description of plants by Eq. (1) is a very common one. However, there are problems in which the evolution-in-time of their variables is most naturally described by means of matrix differential equations. To make this more precise, consider a system whose state variables are x_{ij}, with $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, and whose control variables are $u_{\alpha\beta}$, with $\alpha = 1, 2, \cdots, r$ and $\beta = 1, 2, \cdots, q$. In such problems, one may think of the "state matrix" $\mathbf{X}(t)$, whose elements are the state variables $x_{ij}(t)$, and of the "control matrix" $\mathbf{U}(t)$, whose elements are the control variables $u_{\alpha\beta}(t)$; these are assumed to be related by the matrix differential equation

$$\dot{\mathbf{X}}(t) = \mathbf{F}[\mathbf{X}(t), \mathbf{U}(t), t] \tag{2}$$

where $\mathbf{F}(\cdot)$ is a matrix-valued function of its arguments.

As an example of a system with this type of description, consider a linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{v}(t) \tag{3}$$

where $\mathbf{v}(t)$ is a white-noise process with zero mean and covariance

$$E\{\mathbf{v}(t)\mathbf{v}'(\tau)\} = \delta(t - \tau)\mathbf{Q}(t). \tag{4}$$

If we denote by $\mathbf{\Sigma}(t)$ the covariance matrix of the state vector $\mathbf{x}(t)$, i.e.

$$\mathbf{\Sigma}(t) = E\{\mathbf{x}(t)\mathbf{x}'(t)\}, \tag{5}$$

then it can be shown that $\mathbf{\Sigma}(t)$ satisfies the linear matrix differential equation

$$\dot{\mathbf{\Sigma}}(t) = \mathbf{A}(t)\mathbf{\Sigma}(t) + \mathbf{\Sigma}(t)\mathbf{A}'(t) + \mathbf{Q}(t) \tag{6}$$

which is in the form of Eq. (2). Indeed, there have been some applications of the matrix minimum principle to problems of filtering, control, and signal design (see references *2–7*). In these types of problems one is interested in minimizing a scalar-valued function of the covariance matrix $\mathbf{\Sigma}(t)$ and the "control variables" are some of the elements of the matrix $\mathbf{A}(t)$ and/or $\mathbf{Q}(t)$.

If the system equations are naturally given by Eq. (2), it is easy to visualize an optimization problem. For example, consider a fixed-terminal time-optimization problem with a cost functional

$$J(\mathbf{U}) = K[\mathbf{X}(T)] + \int_{t_0}^{T} L[\mathbf{X}(t), \mathbf{U}(t), t]\, dt \tag{7}$$

where $K[\cdot]$ and $L[\cdot]$ are scalar-valued functions of their argument. One may seek the optimal control-matrix $\mathbf{U}^*(t)$, which may be constrained by

$$\mathbf{U}^*(t) \in \Omega \tag{8}$$

which minimizes the cost functional $\mathbf{J}(\mathbf{U})$.

It should be clear that the tools are available to tackle this optimization problem. After all, one can decompose Eq. (2) into a set of first order equations

$$\dot{x}_{ij}(t) = f_{ij}[\mathbf{X}(t), \mathbf{U}(t), t] \tag{9}$$

and proceed with the application of the familiar minimum principle. However, an excessive number of equations result and it may become almost impossible to determine any structure and properties of the solution. It is this complication which has provided the motivation for dealing with problems involving the time-evolution of matrices by constructing a systematic notational approach.

The first step towards this goal is to realize that the set of all, say, $n \times m$ real matrices forms a linear vector space with well-defined operations of addition and multiplication. Denote this vector space by S_{nm}. Then, it is possible to define an inner product in this space. Thus, if $\mathbf{A}$ and $\mathbf{B}$ are $n \times m$ matrices, i.e. $\mathbf{A} \in S_{nm}$, $\mathbf{B} \in S_{nm}$, their inner product is defined by the trace operation

$$(\mathbf{A}, \mathbf{B}) = \operatorname{tr}[\mathbf{A}\mathbf{B}'] = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij}. \tag{10}$$

It is trivial to verify that Eq. (10) indeed defines an inner product. Using this notation, one can form the Hamiltonian function for the optimization problem. First, note that if $p_{ij}(t)$ is the costate variable associated with $x_{ij}(t)$, then the Hamiltonian must take the form

$$H = L[\mathbf{X}(t), \mathbf{U}(t), t] + \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{x}_{ij}(t) p_{ij}(t). \tag{11}$$

Using Eq. (10), it follows that the Hamiltonian can be written as

$$H = L[\mathbf{X}(t), \mathbf{U}(t), t] + \operatorname{tr}[\dot{\mathbf{X}}(t)\mathbf{P}'(t)], \tag{12}$$

where $\mathbf{P}(t)$ is the costate matrix associated with the state matrix $\mathbf{X}(t)$, in the sense that the costate variable $p_{ij}(t)$ is the ijth element of $\mathbf{P}(t)$.

Using the notation of Athans and Falb (*8*), it is known that the costate variables satisfy the differential equations

$$\dot{p}_{ij}(t) = -\frac{\partial H}{\partial x_{ij}(t)}. \tag{13}$$

This type of equation leads to the definition of the so-called *gradient matrix* (see *9*). Indeed it may be argued that the use of gradient matrices for purely manipulatory purposes is the key concept that makes the use of the matrix minimum principle suitable and straightforward.

A gradient matrix is defined as follows: *Suppose that* $f(\mathbf{X})$ *is a scalar-valued function of the elements* x_{ij} *of* $\mathbf{X}$. *Then the gradient matrix of* $f(\mathbf{X})$ *is denoted by*

$$\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} \tag{14}$$

and it is a matrix whose ijth *element is simply given by*

$$\left[\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}}\right]_{ij} = \frac{\partial f(\mathbf{X})}{\partial x_{ij}}. \tag{15}$$

A brief table of some gradient matrices is given in Appendix A.

Using the notation of the gradient matrix, it is readily seen that Eq. (13) can be written as

$$\dot{\mathbf{P}}(t) = -\frac{\partial H}{\partial \mathbf{X}(t)} \tag{16}$$

since the Hamiltonian H is a scalar-valued function.

Once this notation has been established, one can apply all the known necessary conditions for optimality for vector-type problems to the equivalent statements for the matrix-type problems. In the following section, the necessary conditions for optimality are stated for the fixed-time optimization problem with terminal cost.

2. THE MATRIX MINIMUM PRINCIPLE (CONTINUOUS TIME)

Consider a system with "state matrix" $\mathbf{X}(t)$, "control matrix" $\mathbf{U}(t) \in \Omega$, described by the matrix differential equation

$$\dot{\mathbf{X}}(t) = \mathbf{F}[\mathbf{X}(t), \mathbf{U}(t), t]; \qquad \mathbf{X}(t_0) = \mathbf{X}_0 \tag{17}$$

Consider the cost functional

$$J = K[\mathbf{X}(T)] + \int_{t_0}^{T} L[\mathbf{X}(t), \mathbf{U}(t), t]\, dt; \qquad T \text{ fixed} \tag{18}$$

where $K[\cdot]$ and $L[\cdot]$ are scalar-valued functions of their argument satisfying the usual differentiability conditions.

Let $\mathbf{P}(t)$ denote the costate matrix. Define the scalar Hamiltonian function H by

$$\begin{aligned} H[\mathbf{X}(t), \mathbf{P}(t), t, \mathbf{U}(t)] = {} & L[\mathbf{X}(t), \mathbf{U}(t), t] \\ & + \operatorname{tr}[\mathbf{F}(\mathbf{X}(t), \mathbf{U}(t), t)\mathbf{P}'(t)] \end{aligned} \tag{19}$$

If $\mathbf{U}^*(t)$ is the optimal control, in the sense that it minimizes J, and if $\mathbf{X}^*(t)$ is the corresponding state, then there exists a costate matrix $\mathbf{P}^*(t)$ such that the following conditions hold:

(*i*) *Canonical Equations*

$$\dot{\mathbf{X}}^*(t) = \left.\frac{\partial H}{\partial \mathbf{P}(t)}\right|_* = \mathbf{F}[\mathbf{X}^*(t), \mathbf{U}^*(t), t] \tag{20}$$

$$\begin{aligned} \dot{\mathbf{P}}^*(t) = -\left.\frac{\partial H}{\partial \mathbf{X}(t)}\right|_* = {} & -\frac{\partial}{\partial \mathbf{X}^*(t)} L[\mathbf{X}^*(t), \mathbf{U}^*(t), t] \\ & - \frac{\partial}{\partial \mathbf{X}^*(t)} \operatorname{tr}[\mathbf{F}(\mathbf{X}^*(t), \mathbf{U}^*(t), t)\mathbf{P}^{*\prime}(t)] \end{aligned} \tag{21}$$

(*ii*) *Boundary Conditions*

At the initial time

$$\mathbf{X}^*(t_0) = \mathbf{X}_0 \tag{22}$$

At the terminal time (transversality conditions)

$$\mathbf{P}^*(t) = \frac{\partial}{\partial \mathbf{X}^*(T)} K[\mathbf{X}^*(T)] \tag{23}$$

(*iii*) *Minimization of the Hamiltonian*

$$H[\mathbf{X}^*(t), \mathbf{P}^*(t), t, \mathbf{U}^*(t)] \leqq H[\mathbf{X}^*(t)\mathbf{P}^*(t), t, \mathbf{U}] \tag{24}$$

for every $\mathbf{U} \in \Omega$ and for each $t \in [t_0, T]$.

Note that if $\mathbf{U}(t)$ is unconstrained, then Eq. (24) implies the necessary condition

$$\left.\frac{\partial H}{\partial \mathbf{U}(t)}\right|_* = \mathbf{0}; \tag{25}$$

i.e., the gradient matrix of the Hamiltonian with respect to the control matrix $\mathbf{U}$ must vanish.

3. THE MATRIX MINIMUM PRINCIPLE (DISCRETE TIME)

There are problems for which the evolution of the pertinent variables is most naturally described by a set of matrix difference equations. For such problems, it is possible to extend the results of the "vector" discrete minimum principle (see references *10–12*) to obtain the equivalent form of the discrete matrix minimum principle.

Consider the discrete optimization problem defined by a system of matrix difference equations

$$\mathbf{X}_{k+1} - \mathbf{X}_k = \mathbf{F}_k(\mathbf{X}_k, \mathbf{U}_k); \qquad k = 0, 1, \cdots, N-1 \tag{26}$$

with $\mathbf{U}_k \in \Omega$, $\mathbf{X}_k \in S_{nm}$ for all k, and $\mathbf{U}_k \in S_{\alpha\beta}$. Consider the scalar cost functional

$$J = K(\mathbf{X}_N) + \sum_{k=0}^{N-1} L_k(\mathbf{X}_k, \mathbf{U}_k). \tag{27}$$

It is assumed that $\mathbf{F}_k(\cdot)$, $K(\cdot)$, and $L_k(\cdot)$ satisfy the conditions required by the discrete minimum principle.

Define the Hamiltonian function

$$H(\mathbf{X}_k, \mathbf{P}_{k+1}, \mathbf{U}_k) \triangleq L_k(\mathbf{X}_k, \mathbf{U}_k) + \operatorname{tr}[\mathbf{F}_k(\mathbf{X}_k, \mathbf{U}_k)\mathbf{P}'_{k+1}] \tag{28}$$

where $\mathbf{P}_k$ is the costate matrix.

If $\mathbf{U}_k^*$, $k = 0, 1, \cdots, N-1$ is the optimal control and $\mathbf{X}_k^*$, $k = 0, 1, \cdots, N$, is the optimal state, then the discrete matrix minimum principle states that there exists a costate matrix $\mathbf{P}_k^*$, $k = 0, 1, \cdots, N$, such that the following relations hold:

(*i*) *Canonical Equations*

$$\mathbf{X}_{k+1}^* - \mathbf{X}_k^* = \left.\frac{\partial H}{\partial \mathbf{P}_{k+1}}\right|_* = \mathbf{F}_k(\mathbf{X}_k^*, \mathbf{U}_k^*) \tag{29}$$

$$\mathbf{P}_{k+1}^* - \mathbf{P}_k^* = -\left.\frac{\partial H}{\partial \mathbf{X}_k}\right|_* \tag{30}$$

(*ii*) *Boundary Conditions*

At the initial "time" ($k = 0$)

$$\mathbf{X}_0^* = \mathbf{X}_0 \tag{31}$$

At the terminal "time" ($k = N$)

$$\mathbf{P}_N^* = \frac{\partial}{\partial \mathbf{X}_N^*} K(\mathbf{X}_N^*) \tag{32}$$

(*iii*) *Minimization of the Hamiltonian*

For every $\mathbf{U} \in \Omega$ and each $k = 0, 1, \cdots, N-1$

$$H(\mathbf{X}_k^*, \mathbf{P}_{k+1}^*, \mathbf{U}_k^*) \leqq H(\mathbf{X}_k^*, \mathbf{P}_{k+1}^*, \mathbf{U}). \tag{33}$$

If the $\mathbf{U}_k$ are unconstrained, then Eq. (33) yields the necessary condition

$$\left.\frac{\partial H}{\partial \mathbf{U}_k}\right|_* = \mathbf{0}. \tag{34}$$

4. JUSTIFICATION OF THE MATRIX MINIMUM PRINCIPLE

The extension of the vector minimum principle to the matrix case is straightforward. From a theoretical point of view, it hinges on the existence of a mapping relating the set of $n \times m$ real matrices to the set of (nm)-dimensional vectors.

As before, let S_{nm} denote the set of all real $n \times m$ matrices. Let $R_{(nm)}$ denote the (nm)-dimensional Euclidean vector space. Define a mapping ψ from S_{nm} into $R_{(nm)}$,

$$\psi: S_{nm} \rightarrow R_{(nm)}, \tag{35}$$

so that if $\mathbf{X} \in S_{nm}$ is the matrix

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{bmatrix}, \tag{36}$$

then the image $\mathbf{x} \in R_{(nm)}$ of $\mathbf{X}$ under the mapping is the (nm)-dimensional column vector

$$\mathbf{x} = \begin{bmatrix} x_{11} \\ x_{12} \\ \cdots \\ x_{1m} \\ x_{21} \\ x_{22} \\ \cdots \\ x_{2m} \\ \cdots \\ \cdots \\ x_{nm} \end{bmatrix} = \psi(\mathbf{X}).$$

It is easy to verify that

(1) $\psi(\cdot)$ is a linear mapping,

(2) $\psi(\cdot)$ is one-to-one and onto, hence ψ^{-1} exists,

(3) $\psi(\cdot)$ preserves the inner product because if $\mathbf{X}, \mathbf{Y} \in S_{nm}$ and $\mathbf{x}, \mathbf{y} \in R_{(nm)}$ so that $\mathbf{x} = \psi(\mathbf{X})$, $\mathbf{y} = \psi(\mathbf{Y})$, then the inner product $(\mathbf{X}, \mathbf{Y})$ in S_{nm} is

$$(\mathbf{X}, \mathbf{Y}) = \operatorname{tr}[\mathbf{X}\mathbf{Y}'] = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} y_{ij},$$

while the inner product in $R_{(nm)}$ is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} y_{ij}$$

so that

$$\langle \psi(\mathbf{X}), \psi(\mathbf{Y}) \rangle = (\mathbf{X}, \mathbf{Y}).$$

Thus, the two spaces S_{nm} and $R_{(nm)}$ are algebraically and topologically equivalent.

In the continuous time case, one starts from the matrix differential equation

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, \mathbf{U}, t).$$

Through the mapping ψ, this equation becomes

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{U}; t).$$

Similarly, the integrand of the cost functional $L[\mathbf{X}, \mathbf{U}, t]$ is changed into $L[\mathbf{x}, \mathbf{U}, t]$. Then, by the ordinary vector minimum principle, there is a costate vector $\mathbf{p} \in R_{(nm)}$ associated with $\mathbf{x} \in R_{(nm)}$. Let

$$\mathbf{p} = \begin{bmatrix} p_{11} \\ p_{12} \\ \cdots \\ p_{1m} \\ p_{21} \\ p_{22} \\ \cdots \\ p_{2m} \\ \cdots \\ \cdots \\ p_{nm} \end{bmatrix}.$$

Then the Hamiltonian function in the vector case is

$$H = L(\mathbf{x}, \mathbf{U}, t) + \langle \dot{\mathbf{x}}, \mathbf{p} \rangle.$$

Since $\psi^{-1}(\cdot)$ exists, one can find a unique costate matrix $\mathbf{P} \in S_{nm}$

$$\mathbf{P} = \psi^{-1}(\mathbf{p})$$

so that the Hamiltonian H can be written as

$$H = L(\mathbf{X}, \mathbf{U}, t) + (\dot{\mathbf{X}}, \mathbf{P})$$

in the matrix case. Thus, the fact that ψ preserves the inner product (involved in the definition of the Hamiltonian), coupled with the specific definition of the gradient matrices, yields the matrix minimum principle in the continuous-time case.

Caution. If $\mathbf{X}$ is constrained to be symmetric, then the mapping $\psi(\cdot)$ is *not* invertible. In this case, the definitions of the gradient matrices and the formulae of Appendix A are *not* valid so that the statements in Sections 2 and 3 must be modified in order to obtain the correct answers.

5. APPLICATION TO A LINEAR CONTROL PROBLEM

In this section the matrix minimum principle is used to determine the solution to the simple optimal linear regulator problem. Consider a linear time-varying system with state vector $\mathbf{x}(t)$ and control vector $\mathbf{u}(t)$ related by the vector differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{37}$$

where $\mathbf{A}(t)$ is an $n \times n$ matrix and $\mathbf{B}(t)$ an $n \times r$ matrix. Consider the quadratic cost functional

$$J = \int_{t_0}^{T} [\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt, \tag{38}$$

where $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are symmetric positive-definite matrices. The standard optimization problem is to find the control $\mathbf{u}(t)$, $t_0 \leqq t \leqq T$, so as to minimize the cost functional J.

Instead of dealing with this standard problem, consider the following variation: Suppose that one imposes the *constraint* that the control $\mathbf{u}(t)$ be generated by using a linear time-varying feedback law of the form

$$\mathbf{u}(t) = -\mathbf{G}(t)\mathbf{x}(t), \tag{39}$$

where $\mathbf{G}(t)$ is an $r \times n$ time-varying "gain" matrix (the elements of

$\mathbf{G}(t)$ specify the time-varying feedback gains which multiply the appropriate state variables). In this case, the system satisfies the closed-loop equation

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}(t)]\mathbf{x}(t) \tag{40}$$

and the cost functional J reduces to

$$J = \int_{t_0}^{T} \mathbf{x}'(t)[\mathbf{Q}(t) + \mathbf{G}'(t)\mathbf{R}(t)\mathbf{G}(t)]\mathbf{x}(t)\, dt. \tag{41}$$

To complete the transformation of the problem into the framework required by the matrix minimum principle, define the $n \times m$ "state matrix" $\mathbf{X}(t)$ as the outer vector product of the state vector $\mathbf{x}(t)$ with itself; i.e.,

$$\mathbf{X}(t) \triangleq \mathbf{x}(t)\mathbf{x}'(t). \tag{42}$$

Noting that

$$\mathbf{x}'(t)\mathbf{x}(t) = \operatorname{tr}[\mathbf{X}(t)] \tag{43}$$

$$\mathbf{x}'(t)\mathbf{F}(t)\mathbf{x}(t) = \operatorname{tr}[\mathbf{F}(t)\mathbf{X}(t)] = \operatorname{tr}[\mathbf{X}(t)\mathbf{F}(t)], \tag{44}$$

it follows from Eqs. (42) and (40) that

$$\begin{aligned}\dot{\mathbf{X}}(t) &= \dot{\mathbf{x}}(t)\mathbf{x}'(t) + \mathbf{x}(t)\dot{\mathbf{x}}'(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}(t)]\mathbf{x}(t)\mathbf{x}'(t)\\ &\quad + \mathbf{x}(t)\mathbf{x}'(t)[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}(t)]'\end{aligned} \tag{45}$$

so that the state matrix $\mathbf{X}(t)$ satisfies the linear matrix differential equation

$$\dot{\mathbf{X}}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}(t)]\mathbf{X}(t) + \mathbf{X}(t)[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}(t)]' \tag{46}$$

with the initial condition

$$\mathbf{X}(t_0) = \mathbf{x}(t_0)\mathbf{x}'(t_0). \tag{47}$$

The cost functional J reduces to

$$J = \int_{t_0}^{T} \operatorname{tr}[(\mathbf{Q}(t) + \mathbf{G}'(t)\mathbf{R}(t)\mathbf{G}(t))\mathbf{X}(t)]\, dt \tag{48}$$

The system (46) and cost functional (48) are in the form required to use the matrix minimum principle. So, let $\mathbf{P}(t)$ be the $n \times n$ costate matrix associated with $\mathbf{X}(t)$. The Hamiltonian function H for this problem is[1]

$$H = \operatorname{tr}[\mathbf{QX}] + \operatorname{tr}[\mathbf{G'RGX}] + \operatorname{tr}[\mathbf{AXP'}] - \operatorname{tr}[\mathbf{BGXP'}] \\ + \operatorname{tr}[\mathbf{XA'P'}] - \operatorname{tr}[\mathbf{XG'B'P'}]. \tag{49}$$

The canonical equations yield (using the gradient matrix formulae of Appendix A)

$$\dot{\mathbf{X}} = \frac{\partial H}{\partial \mathbf{P}} = [\mathbf{A} - \mathbf{BG}]\mathbf{X} + \mathbf{X}[\mathbf{A} - \mathbf{BG}]' \tag{50}$$

$$\dot{\mathbf{P}} = -\frac{\partial H}{\partial \mathbf{X}} = -\mathbf{Q} - \mathbf{G'RG} - [\mathbf{A} - \mathbf{BG}]'\mathbf{P} - \mathbf{P}[\mathbf{A} - \mathbf{BG}]. \tag{51}$$

The boundary conditions are

$$\mathbf{X}(t_0) = \mathbf{x}(t_0)\mathbf{x}'(t_0); \qquad \mathbf{P}(T) = \mathbf{0}. \tag{52}$$

Since $\mathbf{G}$ is unconstrained, it is necessary that

$$\mathbf{0} = \frac{\partial H}{\partial \mathbf{G}} = \mathbf{RGX}' + \mathbf{RGX} - \mathbf{B'PX}' - \mathbf{B'P'X}. \tag{53}$$

Note that both $\mathbf{X}(t)$ and $\mathbf{P}(t)$ are symmetric. To see this, note that the solution of (50) is

$$\mathbf{X}(t) = \mathbf{\Phi}(t, t_0)\mathbf{X}(t_0)\mathbf{\Phi}'(t, t_0), \tag{54}$$

where $\mathbf{\Phi}(t, t_0)$ is the transition matrix of $[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}(t)]$. The symmetry of $\mathbf{X}(t)$ follows from (54) and the symmetry of $\mathbf{X}(t_0)$. A similar argument can be used to establish the symmetry of $\mathbf{P}(t)$. These symmetry properties and Eq. (53) yield

$$[\mathbf{R}(t)\mathbf{G}(t) - \mathbf{B}'(t)\mathbf{P}(t)]\mathbf{X}(t) = \mathbf{0}. \tag{55}$$

If this equation is to hold for all $\mathbf{X}(t)$,[2] then one deduces

$$\mathbf{G}(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t). \tag{56}$$

To completely specify the gain matrix $\mathbf{G}(t)$, one must determine the costate matrix $\mathbf{P}(t)$. By substituting Eq. (56) into Eq. (51), one finds that the costate matrix $\mathbf{P}(t)$ is the solution of the familiar Riccati matrix differential equation

[1] The time dependence is suppressed for simplicity.

[2] This is the same argument that one uses in the vector case to obtain the feedback solution. See *8*, p. 761.

$$\begin{aligned}\dot{\mathbf{P}}(t) = &-\mathbf{P}(t)\mathbf{A}(t)\\ &- \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t) - \mathbf{Q}(t)\end{aligned} \tag{57}$$

with the boundary condition

$$\mathbf{P}(T) = \mathbf{0}. \tag{58}$$

It should be clear that the necessary conditions provided by the matrix minimum principle yield the same answer that one would obtain in the vector formulation. It is, of course, well known that the answer is indeed the unique optimal one.

The fact that the costate matrix $\mathbf{P}(t)$ is the solution of the Riccati equation sheds some light in its physical interpretation. If, as required by the Hamilton–Jacobi–Bellman theory, we view the costate matrix as the gradient matrix of the cost with respect to the state, i.e.,

$$\mathbf{P}(t) = \frac{\partial J}{\partial \mathbf{X}(t)}, \tag{59}$$

it is evident that the Riccati equation defines the evolution of the partial derivatives $\partial J/\partial x_{ij}(t)$ for $t \in [t_0, T]$. This conclusion cannot be reached as readily in the vector formulation of the problem.

6. CONCLUSIONS

It has been shown that systems described by matrix differential and difference equations can be optimized by the matrix version of the minimum principle of Pontryagin. The definition of the gradient matrix of a scalar-valued function of a matrix facilitates the manipulation of the necessary conditions for optimality, as illustrated by the problem of optimizing the gains of a linear system.

APPENDIX A. A PARTIAL LIST OF GRADIENT MATRICES

The formulae appearing below have been calculated in the unpublished report by Athans and Schweppe (reference *9*). Some of them have also been calculated by Kleinman, using a different approach (Appendix F of reference *5*). The interested reader should consult these reports for details. The results are stated in this appendix for the sake of reference. The calculations involved are straightforward but lengthy.

In the formulae below, $\mathbf{X}$ is an $n \times m$ matrix. The reader is cautioned that the formulae are not valid if the elements x_{ij} of $\mathbf{X}$ are not independent.

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{X}] = \mathbf{I} \tag{A.1}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{AX}] = \mathbf{A}' \tag{A.2}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{AX}'] = \mathbf{A} \tag{A.3}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{AXB}] = \mathbf{A}'\mathbf{B}' \tag{A.4}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{AX}'\mathbf{B}] = \mathbf{BA} \tag{A.5}$$

$$\frac{\partial}{\partial \mathbf{X}'} \operatorname{tr}[\mathbf{AX}] = \mathbf{A} \tag{A.6}$$

$$\frac{\partial}{\partial \mathbf{X}'} \operatorname{tr}[\mathbf{AX}'] = \mathbf{A}' \tag{A.7}$$

$$\frac{\partial}{\partial \mathbf{X}'} \operatorname{tr}[\mathbf{AXB}] = \mathbf{BA} \tag{A.8}$$

$$\frac{\partial}{\partial \mathbf{X}'} \operatorname{tr}[\mathbf{AX}'\mathbf{B}] = \mathbf{A}'\mathbf{B}' \tag{A.9}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{XX}] = 2\mathbf{X}' \tag{A.10}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{XX}'] = 2\mathbf{X} \tag{A.11}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{X}^n] = n(\mathbf{X}^{n-1})' \tag{A.12}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{AX}^n] = \left(\sum_{i=0}^{n-1} \mathbf{X}^i\mathbf{AX}^{n-1-i}\right)' \tag{A.13}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{AXBX}] = \mathbf{A}'\mathbf{X}'\mathbf{B}' + \mathbf{B}'\mathbf{X}'\mathbf{A}' \tag{A.14}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{AXBX}'] = \mathbf{A}'\mathbf{X}'\mathbf{B}' + \mathbf{AXB} \tag{A.15}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[e^{\mathbf{X}}] = e^{\mathbf{X}'} \tag{A.16}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{X}^{-1}] = -(\mathbf{X}^{-1}\mathbf{X}^{-1})' = -(\mathbf{X}^{-2})' \tag{A.17}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}[\mathbf{A}\mathbf{X}^{-1}\mathbf{B}] = -(\mathbf{X}^{-1}\mathbf{B}\mathbf{A}\mathbf{X}^{-1})' \tag{A.18}$$

$$\frac{\partial}{\partial \mathbf{X}} \det[\mathbf{X}] = (\det[\mathbf{X}])(\mathbf{X}^{-1})' \tag{A.19}$$

$$\frac{\partial}{\partial \mathbf{X}} \log \det[\mathbf{X}] = (\mathbf{X}^{-1})' \tag{A.20}$$

$$\frac{\partial}{\partial \mathbf{X}} \det[\mathbf{A}\mathbf{X}\mathbf{B}] = (\det[\mathbf{A}\mathbf{X}\mathbf{B}])(\mathbf{X}^{-1})' \tag{A.21}$$

$$\frac{\partial}{\partial \mathbf{X}} \det[\mathbf{X}'] = \frac{\partial}{\partial \mathbf{X}} \det[\mathbf{X}] = (\det[\mathbf{X}])(\mathbf{X}^{-1})' \tag{A.22}$$

$$\frac{\partial}{\partial \mathbf{X}} \det[\mathbf{X}^{n}] = n(\det[\mathbf{X}])^{n}(\mathbf{X}^{-1})' \tag{A.23}$$

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